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Vertical Motion Control of a Hopping Robot *

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Abstract

In this paper we present a new approach for vertical motion feedback control of a hopping robot. The task of the feedback control is to stabilize the robotic system to a desired limit cycle under existing constraints on the controls. Unlike most previous methods, the approach presented here does not rely on the construction of the Poincaré map, which simplifies the subsequent analysis, but can be difficult or impossible to obtain analytically. The method can also be applied to more complicated, possibly nonlinear models of legged robots, as well as other variable structure systems.

Keywords: legged locomotion, mobile robots, variable structure systems, stabilization, limit cycles.

1 Introduction

This paper describes a novel approach to the feedback control of the vertical motion of a hopping robot. This control is modulating the response of a spring-mass system, formed by the robot's body mass and a compliant leg. Apart from forward speed control, stable control of the vertical oscillation is central to any successful implementation of dynamically stable legged locomotion.

Legged robots are intermittent dynamical systems and fall into the category of variable structure systems, i.e. systems characterized by different mathematical descriptions in non-overlapping regions of the state space. The analysis of such systems has long been recognized as complex; interesting dynamical behaviour including possible existence of higher order limit cycles, bifurcations and strange attractors has been demonstrated [5, 3, 9], based on simplified models of Raibert's [8] one-legged robots and his control laws.

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Due to the enormous success of Raibert's pioneering experimental work, his models have become a standard for most of the past research on further controller development and stability analysis. Fortunately, since an impulsive control input is applied, and the vertical hopper dynamics are essentially second order, the analytical integration of the stance phase dynamics is possible. This in turn permits the derivation of a scalar Poincaré map, for which a bounty of analytical techniques have been developed. This approach is followed in [3, 4, 5, 9].

Analysis of the continuous dynamics is difficult, even for second order systems, since most of the standard mathematical results from nonlinear systems theory do not apply, due to the variable structure of the system. The approaches have been varied but apply some form of perturbation theory, as for example in [2, 7]. In both Poincaré and perturbation approaches, model simplifications allow for successful analysis, but can lead to serious inaccuracies. To our knowledge, none of the above stability results have been experimentally validated.

Our past experimental work on electrically actuated legged robots has revealed two important differences to the standard model used in the past based on Raibert's robots. First, the need for transmissions necessitates the inclusion of the actuator dynamics, which makes the simplest system fourth order. Second, due to actuator torque constraints, impulsive inputs are out of question, and torque must be applied throughout the stance phase. As a result, integrating the equations of motion analytically is intractable, and therefore the Poincaré map is not available. We investigate the problem of stabilization to limit cycles from an entirely different point of view. Instead of concentrating on finding all the possible fixed points of a Poincaré map for a closed loop system with an imposed controller structure, we start by designing a reference trajectory which realises a limit cycle of a desired size. A systematic method is proposed next for the construction of feedback controllers which en-

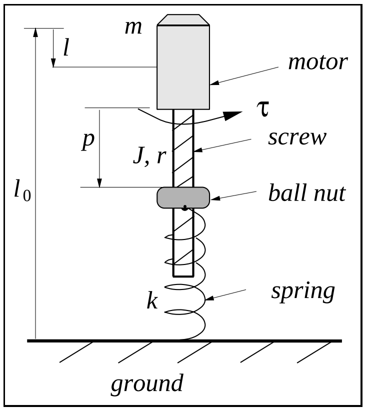

Figure 1: *The simplified model of the Vertical Hopper.*

sure global convergence of the closed loop system trajectories to the reference limit cycle. The discretized version of the proposed feedback controller leads to piece-wise linear control which is particularly convenient when control constraints have to be incorporated. The advantages of our approach and the contributions of this paper are as follows:

- A novel method is presented for the construction of feedback controllers which stabilise a class of variable structure systems to reference limit cycles.
- There is considerable freedom in the design of the reference limit cycle to accommodate additional design objectives or control constraints.
- The proposed feedback law follows from a simple phase plane analysis of a reduced order model and its convergence is intuitively clear.
- The analysis of the feedback law is based on a globally attractive reference limit cycle and does not require the construction of the Poincaré map nor calculation of its fixed points.
- As a feedback law is used, we expect it to be robust to modelling errors and disturbances.

2 The model and the control problem

The physical model of our robot [1] is shown in Figure 1. The body with mass m and height l is attached to an electric motor whose torque τ drives a ball screw with lead r, inertia J and $\alpha = J/r$. The ball nut displaces by p the leg spring with stiffness k. For notational simplicity, and without restriction of generality, we offset the body height l by l_0, the elevation of the body at touchdown. Neglecting the lower leg mass, and assuming that during stance spring forces dominate friction, we obtain the model

Flight phase (valid for $l(t) > 0$) :

$$\ddot{l}(t) = -g \tag{1}$$

Stance phase (valid for $l(t) < 0$) :

$$\ddot{p}(t) = \frac{1}{\alpha}\tau(t) - d[p(t) - l(t)] \tag{2}$$

$$\ddot{l}(t) = c[p(t) - l(t)] - g \tag{3}$$

where g is the gravitational acceleration and

$$c \stackrel{def}{=} \frac{k}{m}, \quad d \stackrel{def}{=} \frac{rk}{\alpha}.$$

This is a variable structure system as the equations of motion are different in the half-spaces to the left and right of $l = 0$. The model is only complete with the initial conditions at the beginning of the i-th stance phase, t_i^s, also called the touchdown time,

$$l(t_i^s) = 0, \quad \dot{l}(t_i^s) = \dot{l}_i^s \tag{4}$$

$$p(t_i^s) = \dot{p}(t_i^s) = 0. \tag{5}$$

From the flight phase equation it follows that $\dot{l}_{i+1}^s = -\dot{l}(t_i^f)$ where t_i^f denotes the takeoff moment in the i-th hopping cycle.

Since the velocity at touchdown is trivially related to the hopping height, the control problem is easier stated in terms of reference velocity. This gives rise to the set point control problem (SPC).

SPC: Find a feedback control strategy in terms of a control torque

$$\tau(x) \in \mathcal{D} \quad \text{for } x \in I\!R^4 \tag{6}$$

such that for any given set of boundary conditions (5) :

$$\dot{l}(t_i^s) \rightarrow \dot{l}_{des} \quad \text{as } i \rightarrow \infty \tag{7}$$

where $x \stackrel{def}{=} [l, \dot{l}, p, \dot{p}]^T$ represents the state of the system in (1 - 3), $\mathcal{D}$ is a set of admissible control values defined by the actuating motor, typically, for given constants $\gamma_i > 0, i = 1, 2$:

$$\mathcal{D} \stackrel{def}{=} \{\ y \in I\!R \ : \ y = \tau(x), \ |\frac{d}{d\dot{p}}\tau(x)| \leq \gamma_1, \ |\frac{d}{dp}\tau(x)| \leq \gamma_2 \ \} \tag{8}$$

and with $\dot{l}_{des}$ being the reference velocity at touchdown which yields the desired hopping height.

The control constraint (8) clearly delimits the "gains" of the control with respect to variables p and $\dot{p}$; if, for example, $\tau(x) = \alpha_1 \dot{p} + \alpha_2 p$ then for the control constraints to be satisfied it must hold that $\alpha_i \leq \gamma_i$, $i = 1, 2$.

3 The feedback control and its analysis

Transformation to an equivalent model
To facilitate controller design it is convenient to derive an equation which directly relates the body height l and the control torque τ. This can be achieved by differentiating (3) twice and substituting $\ddot{l}$ and $\ddot{p}$ from (2) and (3) again,

$$l^{(4)} = c(\ddot{p} - \ddot{l}) = c[\frac{\tau}{\alpha} - (c + d)(p - l) + g].$$

After solving (3) for $(p - l)$ and substituting in the above equation, we obtain the desired equation only in l and τ

$$l^{(4)} = \frac{c}{\alpha}\tau(t) - (c + d)\ddot{l}(t) - dg \tag{9}$$

which, together with (3), initial conditions (4) and the additional higher order initial conditions

$$\ddot{l}(t_i^s) = -g \tag{10}$$
$$l^{(3)}(t_i^s) = -c\dot{l}_i^s \tag{11}$$

describe the stance dynamics.

At this point, it is clear that, in the absence of control constraints, the control torque can always be chosen in such a way as to provide for an essentially arbitrary stance phase dynamics. Note also that the torque can always be expressed as a function of the state vector only, i.e. it does not require measurement of accelerations, by utilizing equation (3) to eliminate second and third order derivatives of l. It is also easy to verify that for any given linear controller $\tau(x)$, analytic calculation of the corresponding Poincaré map for system described by (3) and (9) is impossible. The above reasons motivate the following alternative approach to the construction of feedback control which solves SPC.

Construction of a nominal controller
The aim here is to design a nominal limit cycle trajectory and control during stance which requires the least energy input from the controller during steady state and which is then most likely to satisfy any realistic control constraint during transients as well. This choice would require no actuator movement at all, $p(t) = \dot{p}(t) = \ddot{p}(t) = 0$, for which the nominal controller can be read off directly from (2) as

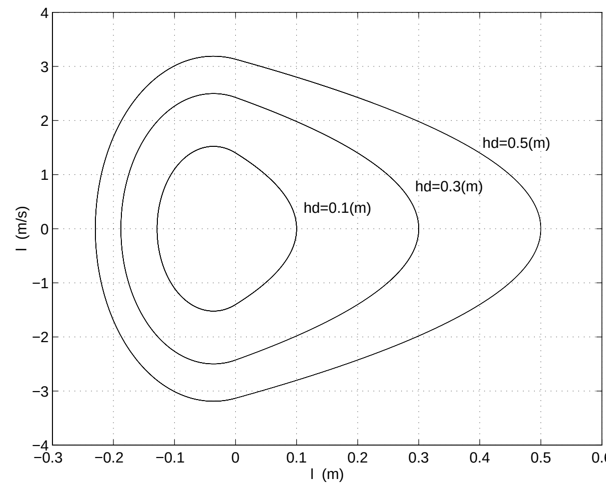

Figure 2: *Desired limit cycles corresponding to each desired hopping height h_d.*

$$\tau = rk(p - l). \tag{12}$$

The result is the simple second order stance phase dynamics of a pure mass-spring system,
Flight phase (valid for $l(t) > 0$) :

$$\ddot{l}(t) = -g \tag{13}$$

Stance phase (valid for $l(t) < 0$) :

$$\ddot{l}(t) = -cl(t) - g \tag{14}$$
$$\text{with} \quad l(0) = 0, \;\; \dot{l}(0) = \dot{l}_{des} < 0 \tag{15}$$

A plot of a few trajectories of the closed loop system described by equations (1-5) together with the feedback law (12) and a few choices of the desired hopping height (different values of the initial condition $\dot{l}_i^s$) are shown in the phase plane $(l, \dot{l})$ in Figure 2.

The phase space trajectories are cyclic. The stance phase is basically modelled as a simple mass spring system under gravity with nonzero initial speed. The solution to differential equation (14) with the initial condition of (15) has an elliptic phase trajectory of the form

$$(l + \frac{g}{c})^2 + \frac{\dot{l}^2}{c} = R^2, \tag{16}$$

$$R^2 \stackrel{def}{=} (\frac{g}{c})^2 + \frac{\dot{l}^2_{des}}{c}. \tag{17}$$

Stabilising feedback control design for unconstrained SPC

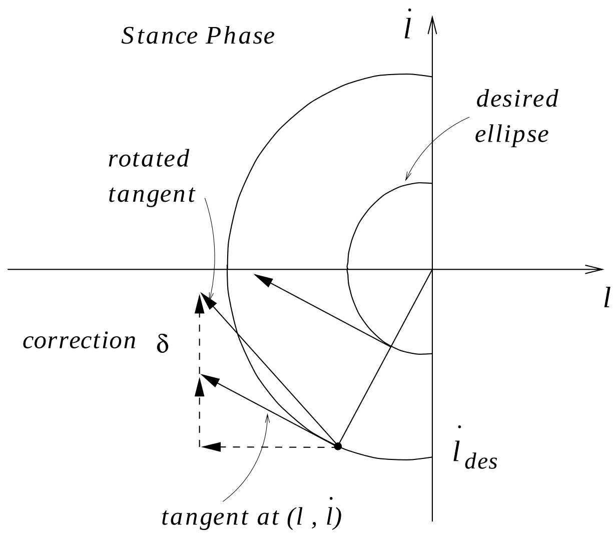

Figure 3: *The correction term forces the system to the desired limit cycle.*

The task here is to design a control law which makes the designed limit cycle globally attractive. The construction is easiest explained for the "reduced order" model, in the phase plane $(l, \dot{l})$, while temporarily discarding the control constraints. Whatever the value of the initial velocity at touchdown, $\dot{l}(t_i^s)$, the control law (12) maintains system motion in a limit cycle whose stance part is described by (16-17) in which $\dot{l}(t_i^s)$ replaces $\dot{l}_{des}$. The tangent vector to the limit cycle at any of its points $(l, \dot{l})$ is $[\dot{l}, -cl - g]$ and represents the direction of system motion in the phase space. From Figure 3 it is clearly seen that if such a point belongs to a limit cycle which is too large (for which $|\dot{l}(t_i^s)| > |\dot{l}_{des}|$) then convergence towards the limit cycle of a desired size would take place if this tangent vector could somehow be rotated to point towards the desired cycle. Such rotation could be realized by introducing a correction $\delta(l, \dot{l})$ to the vertical component of the tangent vector in such a way that

$$\delta(l, \dot{l}) > 0 \quad \text{for} \quad \dot{l} < 0 \tag{18}$$
$$\delta(l, \dot{l}) < 0 \quad \text{for} \quad \dot{l} > 0. \tag{19}$$

Additionally, the above correcting term ought to decrease with the distance to the desired cycle. A possible choice of such a correction, which has the advantage of being smooth is

$$\delta(l, \dot{l}) = -\beta \ \dot{l} \ [(l + \frac{g}{c})^2 + \frac{\dot{l}^2}{c} - R^2] \tag{20}$$

where β is some positive constant to be selected later. A similar analysis can be carried out when the system motion starts in the inside, rather than on the outside of the desired cycle, only to show that (20) is again valid. The tangent to the "corrected" trajectory in the phase plane is then equal to $[\dot{l}, -cl - g + \delta(l, \dot{l})]$. The knowledge of a tangent vector to a phase space trajectory permits immediate reconstruction of a differential equation describing the motion along this trajectory,

$$\ddot{l} = -cl - g + \delta(l, \dot{l}). \tag{21}$$

In the light of the above discussion, it is intuitively clear that (21) together with (13) represent a spiral shaped trajectory which "winds up" to the desired nominal cycle. In fact, employing a Lyapunov type function

$$V(l, \dot{l}) \stackrel{def}{=} [(l + \frac{g}{c})^2 + \frac{\dot{l}^2}{c} - R^2]^2 \tag{22}$$

and using the results in [10], we can prove rigorously, see [6],

Theorem 1 *For every constant $\beta > 0$ the system described by equations (13) and (21) has a unique, globally attractive limit cycle, whose equations of motion are (13 - 15).*

It is straightforward to derive a control law τ such that the closed loop system behaves like (13) and (21). Differentiating (21) twice with δ as in (20) yields

$$\begin{aligned} l^{(4)} &= -2\beta\ddot{l}^3 + \ddot{l}[-c - 6\beta\dot{l}(l + \frac{g}{c}) - \frac{6\beta}{c}\dot{l}\,\ddot{l}] \\ &\quad -\beta l^{(3)}[(l + \frac{g}{c})^2 + \frac{3}{c}\dot{l}^2 - R^2]. \end{aligned} \tag{23}$$

Equating the right hand sides of (23) and (9) gives

$$\begin{aligned} \tau &= \frac{\alpha}{c}\ \{dg - 2\beta\dot{l}^3 \\ &\quad + \ddot{l}[d - 6\beta\dot{l}(l + \frac{g}{c}) - \frac{6\beta}{c}\dot{l}\,\ddot{l}] \\ &\quad -\beta l^{(3)}[(l + \frac{g}{c})^2 + \frac{3}{c}\dot{l}^2 - R^2]\} \end{aligned} \tag{24}$$

which after replacing higher order derivatives $\ddot{l} = c(p - l) - g$ and $l^{(3)} = c(\dot{p} - \dot{l})$, will be only in terms of the state variables $[l, \dot{l}, p, \dot{p}]$. When applied to (9), the control τ will result in a trajectory described by (21) only if (23) integrated twice gives (21). The latter is valid provided that

$$\ddot{l}(0) = -g + \delta(0, \dot{l}(0)) \tag{25}$$
$$l^{(3)}(0) = -c\dot{l}(0) + \dot{\delta}(0, \dot{l}(0)) \tag{26}$$

in which

$$\begin{aligned} \dot{\delta}(0, \dot{l}(0)) &= -\beta\ddot{l}(0) \cdot \\ [(\frac{g}{c})^2 + 3\frac{(\dot{l}(0))^2}{c} - R^2] &- 2\beta\frac{g}{c}\dot{l}^2(0). \end{aligned} \tag{27}$$

Comparing the right hand sides of (25-26) with

$$\ddot{l}(t_i^s) = c(p(t_i^s) - l(t_i^s)) - g, \; l^{(3)}(t_i^s) = c(\dot{z}(t_i^s) - \dot{l}(t_i^s)) \quad (28)$$

at touchdown, and recalling that $\dot{l}_{i+1}^s = -\dot{l}(t_i^f)$, yields new boundary conditions for p and $\dot{p}$,

$$p(t_i^s) = \frac{1}{c}\,\delta(0,\dot{l}_i^s); \quad \dot{p}(t_i^s) = \frac{1}{c}\,\dot{\delta}(0,\dot{l}_i^s). \quad (29)$$

It follows that the actuator states p and $\dot{p}$ have to be reset to those given in (29) during every flight phase. This can be accomplished by employing an additional PID controller during the flight phase.

Provided the latter is insured, equations (2-3), with the control law given by (24) and boundary conditions as in (4) and (29), are equivalent to equations (21) and $p = \delta(l,\dot{l})/c$ with initial conditions $l(0) = 0$ and $\dot{l}(0) = \dot{l}_i^s$.

We then have the following consequence of Theorem 1.

Corollary 1 *For any desired velocity at touchdown the controlled system with τ given by (24) and boundary conditions for p and $\dot{p}$ as in (29) converges globally to the desired limit cycle given in terms of (13-15). The speed of convergence increases with the value of β.*

In conclusion it is worth noting that the approach presented can utilise other models of reference limit cycles and is not limited to the choice of the correcting term δ as in (20). Such flexibility in the design permits, for example, to take account of additional control objectives.

4 Simplifications, control constraints, and simulation

The controller constructed in the previous section is highly nonlinear, requires continuous measurement of the velocity variables $\dot{p}$ and $\dot{l}$ and may not satisfy control constraints. From a practical point of view it is therefore meaningful to look for simplified control laws. We present here only two such possibilities.

First, suppose the correcting term δ is updated only at each touchdown. Hence

$$\delta(l,\dot{l}) = \delta(0,\dot{l}_i^s) = -\beta\,\dot{l}_i^s\,[(\frac{g}{c})^2 + \frac{(\dot{l}_i^s)^2}{c} - R^2] \stackrel{def}{=} \delta_i \quad (30)$$

is piecewise constant and gives a simpler equation replacing (21) with

$$\ddot{l} = -cl - g + \delta_i. \quad (31)$$

Differentiating (31) twice is identical to the second derivative of (14) and confirms that the corresponding controller is linear and identical to (12). Note that the value of the correction term δ has effect on system convergence only through the boundary conditions (29). To summarise, discretization of δ results in the following proportional feedback control law

$$\tau = rk(p - l) \quad (32)$$

with the necessity of resetting the values of p and $\dot{p}$ to

$$p(t_i^s) = \frac{1}{c}\,\delta_i, \;\; \dot{p}(t_i^s) = 0 \quad (33)$$

during each flight phase.

Second, let us also consider the case when only a part of the correction term δ is discretized, for example let δ be given by:

$$\delta(l,\dot{l}) = -\beta\,\dot{l}\,[(\frac{g}{c})^2 + \frac{(\dot{l}_i^s)^2}{c} - R^2] \stackrel{def}{=} -\beta\,\dot{l}\rho_i. \quad (34)$$

The correspondent of equation (23) is now:

$$l^{(4)} = -c\ddot{l} - \beta\rho_i l^{(3)} \quad (35)$$

which leads to a simple PD stabilising feedback controller with initial conditions (29)

$$\tau = -\beta\rho_i l^{(3)} + d\ddot{l} + dg = -\beta\rho_i(\dot{p} - \dot{l}) + rk(p - l). \quad (36)$$

The above two control laws are not only linear and much simpler than (24) but allow to take the control constraints into account. If only $|rk| \leq \gamma_1$ (which is supposed to be guaranteed by the choice of the reference limit cycle itself), then β can be chosen such that $\beta\rho_0 \leq \gamma_2$, hence providing for the satisfaction of the control constraints as stated in SPC. We are also able to state the following convergence result for the simplified laws [6],

Theorem 2 *For every bounded region of initial velocities at touchdown, there exists a constant β for which the control laws (32) or (36), with their corresponding boundary conditions for p and $\dot{p}$ (33), solve the SPC.*

The results of the simulations confirm the effectiveness of the proposed stabilising feedback. The trajectory of Figure 4 corresponds to the continuous time control of (24). Figure 5 demonstrates convergence to the reference limit cycle while employing the PD feedback law (36).

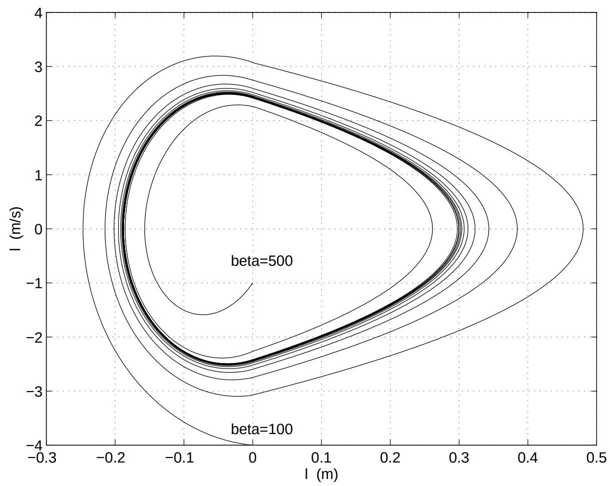

Figure 4: *Spiral trajectories toward the desired limit cycle with different convergence gains β and different initial velocities.*

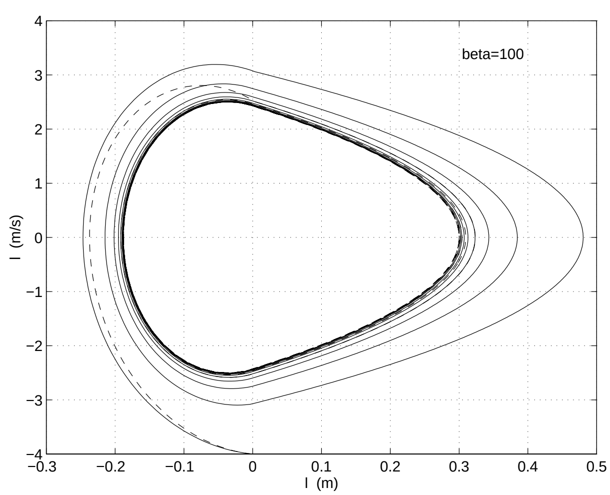

Figure 5: *Comparing the spirals from continuous control scheme (solid line) and discretized control scheme (dashed line).*

5 Conclusion

As confirmed by preliminary experiments, the simple, planar model (1-3) seems to be sufficiently precise for the control law to be implemented on a laboratory prototype of a hopper. Further research will be concerned with improved simplifications of the control and better choices of reference cycles.

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